# Econ 802
## Answers for Second Midterm

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1. (a) The assumptions on Jane's preferences ensure that her preference ordering can be represented by a continuous utility function $u(x)$. The feasible set is $B = \{x \geq 0 : px \leq m\}$ where $p > 0$, $m > 0$. This set is non-empty because it contains the origin. It is closed because it contains all boundary points. It is bounded because there is some finite distance $\varepsilon > 0$ such that all feasible points are closer than $\varepsilon$ to the origin. Therefore the feasible set $B$ is compact. The Weierstrasse Theorem says that a continuous function defined on a non-empty compact set takes on a maximum value somewhere in the set. So yes, Jane's problem has a solution.

(b) Local non-satiation says that for any bundle $x \in X$ and any distance $\varepsilon > 0$, there is some other bundle $y \in X$ with $|x - y| < \varepsilon$ such that $y$ is strictly preferred to $x$. This implies that an optimal bundle must have $px = m$, i.e. it is on the budget line rather than below it. Suppose there is an optimal bundle $x \in X$ with $px < m$. Then we can choose some $\varepsilon > 0$ such that all $y$ with $|x - y| < \varepsilon$ also have $py < m$ so they are feasible. But some $y$ vector that is affordable must be preferred to $x$. This contradicts the optimality of $x$.

Strict quasi-concavity of the utility function means that upper contour sets of the form $\{x \geq 0 : u(x) \geq u\}$ are strictly convex. This implies that the solution to the utility max problem is unique. Suppose instead that $x$ and $x'$ both max $u(x)$ subject to $px \leq m$ with $x \neq x'$. Then construct $x'' = tx + (1-t)x'$ with $0 < t \leq 1$. It is also true that $px'' \leq m$ so $x''$ is feasible. But strict convexity implies that $x''$ is strictly preferred to $x$ and $x'$, which contradicts optimality for these bundles.

(c) Sufficient SOC: $h' \frac{\partial^2 u(x^*)}{\partial x^2} h < 0$ for all $h \neq 0$ such that $p \cdot h = 0$ (or you could say such that $\frac{\partial u(x^*)}{\partial x} \cdot h = 0$). To derive it, expand $u(x)$ around a point $x^*$ that satisfies the FOC: $\frac{\partial u(x^*)}{\partial x} = \lambda p$. This gives

$$u(x) = u(x^*) + \frac{\partial u(x^*)}{\partial x}(x - x^*) + \frac{1}{2}(x-x^*)' \frac{\partial^2 u(x^*)}{\partial x^2}(x - x^*)$$

+ higher order terms.

Define $h = x - x^*$. We are only interested in $x$ such that $px = m$. Local non-satiation says we can ignore $x$ with $px < m$ (they will never be optimal) and the budget constraint says we can ignore $px > m$ (not feasible). So we limit attention to $px = px^* = m$ or $p \cdot h = 0$. We need to be sure that $x^*$ gives a local max of $u(x)$ ie $u(x) < u(x^*)$ for sufficiently small $h$. From FOC we have $\frac{\partial u(x^*)}{\partial x} \cdot h = \lambda p \cdot h = 0$, so we can ignore first order terms. When $h$ is small we can ignore the higher order terms. So a sufficient SOC is $h' \frac{\partial^2 u(x^*)}{\partial x^2} h < 0$ for all $h \neq 0$ such that $p \cdot h = 0$.

The useful thing: when the sufficient SOC holds, we can use the implicit function theorem to show that the Marshallian demands $x(p, m)$ are differentiable functions of the exogenous parameters $(p, m)$.

2. (a) We need to find $e(p, u) = \min \sum_i p_i x_i$
subject to $u(x) \geq u$.

The constraint implies $a_i x_i \geq u$ for all $i = 1 \ldots n$.
An optimum cannot have $a_i x_i > u$ for any $i$ because then it would be possible to reduce expenditure by reducing $x_i$ a little bit without violating the constraint. So

$$a_1 x_1 = a_2 x_2 = \cdots a_n x_n = u \quad \text{or} \quad h_i(p, u) = \frac{u}{a_i} \text{ for all } i$$

Note that the Hicksian demands don't depend on $p$.
Finally, $e(p, u) = \sum_i p_i \frac{u}{a_i} = u \sum_{i=1}^{n} \frac{p_i}{a_i}$.

(b) Invert the expenditure function to get the indirect utility function: $V(p, m) = \dfrac{m}{\sum_i p_i / a_i}$

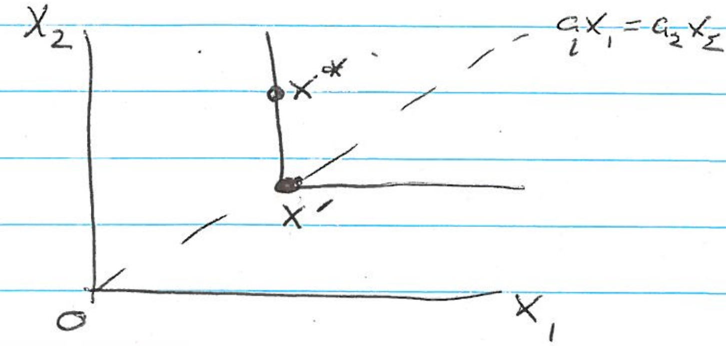
Then use Roy's identity:

$$x_i(p, m) = -\frac{\frac{\partial V(p,m)}{\partial p_i}}{\frac{\partial V(p,m)}{\partial m}} = \frac{-\left[\frac{-m}{(\Sigma_i)^2} \frac{1}{a_i}\right]}{\frac{1}{(\Sigma_i)}} = \frac{m}{a_i} \frac{1}{\left(\sum_i \frac{p_i}{a_i}\right)}$$

for all $i = 1 \ldots n$

(c) For an arbitrary consumption bundle $x$, the inverse demand function gives a price vector $p$ such that $x$ is demanded when the prices are $p$ and income is $m = 1$.

This concept does not make sense with Leontief preferences. Consider the following graph:




At a point like $x^*$, there is no budget line with $p > 0$ such that $x^*$ is utility maximizing.
At a point like $x'$, we can find a price vector such that the budget line passes through $x'$, and therefore $x'$ is utility maximizing. But there are many such price vectors so the inverse demand function is not well-defined.

3. (a) $V(p,m) = \max \{a \ln x_1 + b \ln x_2\}$ subj to $p_1 x_1 + p_2 x_2 = m$

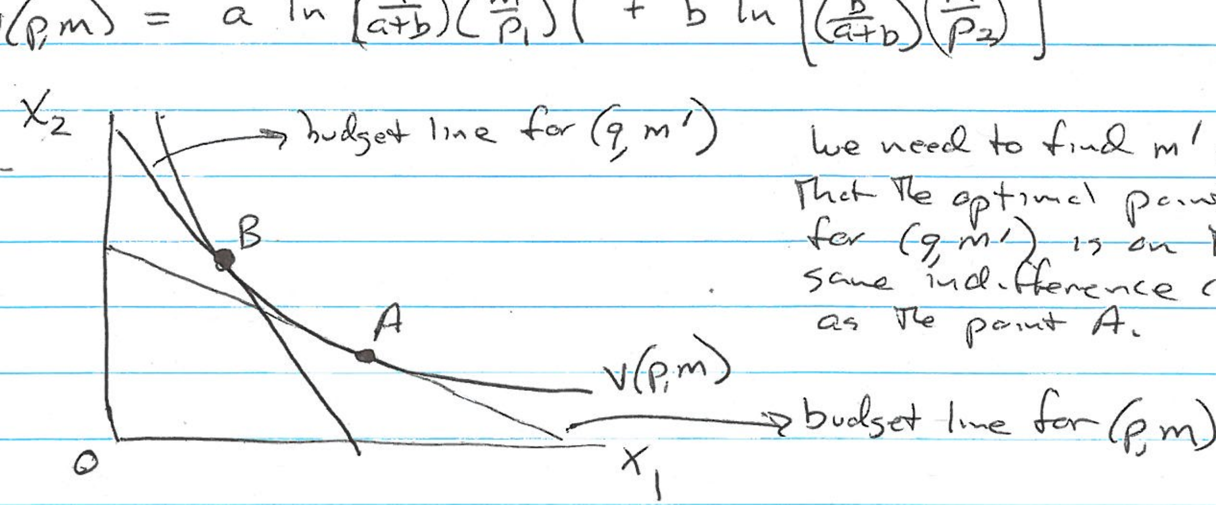
FOC: $\frac{a}{x_1} = \lambda p_1$, $\frac{b}{x_2} = \lambda p_2$ (and the budget constraint)

$\Rightarrow p_1 x_1 = \frac{a}{\lambda}$ ; $p_2 x_2 = \frac{b}{\lambda} \Rightarrow \frac{a}{\lambda} + \frac{b}{\lambda} = m \Rightarrow \frac{1}{\lambda} = \frac{m}{a+b}$
$\Rightarrow \lambda = \frac{a+b}{m}$

$\Rightarrow x_1 = \left(\frac{a}{a+b}\right)\frac{m}{p_1}$ and $x_2 = \left(\frac{b}{a+b}\right)\frac{m}{p_2}$.

So $v(p,m) = a \ln\left[\left(\frac{a}{a+b}\right)\left(\frac{m}{p_1}\right)\right] + b \ln\left[\left(\frac{b}{a+b}\right)\left(\frac{m}{p_2}\right)\right]$

(b)




We need to find $m'$ such that the optimal point B for $(q, m')$ is on the same indifference curve as the point A.

A convenient transformation of utility would be to get rid of the natural logs by working with $e^{V(p,m)}$ instead. Note that this is equivalent to working with the direct utility function $x_1^a x_2^b$ (Cobb-Douglas) instead of $a \ln x_1 + b \ln x_2$. So write indirect utility as

$$\left[\left(\frac{a}{a+b}\right)\left(\frac{m}{p_1}\right)\right]^a \left[\left(\frac{b}{a+b}\right)\left(\frac{m}{p_2}\right)\right]^b,$$

It also does no harm to assume $a+b=1$ (we can always impose this through an increasing transformation of the utility function).

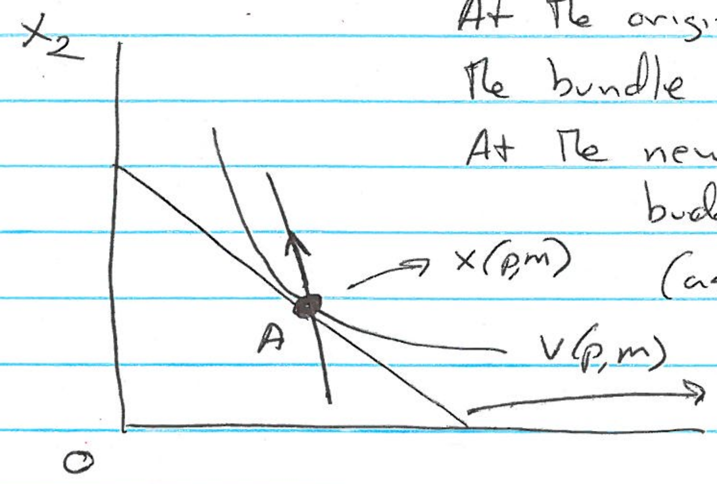
Setting the two utility levels equal gives

$$\left[\frac{am}{p_1}\right]^a \left[\frac{bm}{p_2}\right]^b = \left[\frac{am'}{q_1}\right]^a \left[\frac{bm'}{q_2}\right]^b$$

or using $a+b=1$, $m\left[\left(\frac{a}{p_1}\right)^a\left(\frac{b}{p_2}\right)^b\right] = m'\left(\frac{a}{q_1}\right)^a\left(\frac{b}{q_2}\right)^b$

$$\Rightarrow \boxed{m' = m\left(\frac{q_1}{p_1}\right)^a\left(\frac{q_2}{p_2}\right)^b}$$

(c) She is better off. This is clear from the following graph:




At the original prices and income $(p,m)$, the bundle $x(p,m)$ at point $A$ is optimal. At the new prices $q$, the slope of the budget line will be either steeper (as shown) or flatter.

In either case if income adjusts so that point A is still feasible (the new budget line passes through this point) then it is possible to get to a higher indifference curve by moving along the new budget line. For instance, if the line is steeper, she can move along the line up and to the left as shown.

4. (a) Marshallian: $x_i(p,m) = -\dfrac{\frac{\partial V(p,m)}{\partial p_i}}{\frac{\partial V(p,m)}{\partial m}} = -\dfrac{\frac{\partial w(p)}{\partial p_i} \cdot m}{w(p)}$

Hicksian: $h_i(p,u) = \dfrac{\partial e(p,u)}{\partial p_i} = \dfrac{-u}{w(p)^2} \dfrac{\partial w(p)}{\partial p_i}$

(b) First compute the overall effect:

$$\frac{\partial x_i}{\partial p_i} = -\frac{\frac{\partial^2 w}{\partial p_i^2} m}{w(p)} + \frac{\left(\frac{\partial w(p)}{\partial p_i}\right)^2 m}{w(p)^2}$$

From the Slutsky equation, $\dfrac{\partial x_i}{\partial p_i} = \underbrace{\dfrac{\partial h_i[p, v(p,m)]}{\partial p_i}}_{\text{substitution effect}} - \underbrace{\dfrac{\partial x_i}{\partial m}}_{\text{income effect.}} \cdot \cancel{x}_i$

Substitution effect:

$$\frac{\partial h_i(p,u)}{\partial p_i} = -u\left[\frac{-2}{w(p)^3}\left(\frac{\partial w}{\partial p_i}\right)^2 + \frac{1}{w(p)^2}\frac{\partial^2 w}{\partial p_i^2}\right]$$

To get rid of $u$, replace it with $v(p,m) = w(p)m \Rightarrow$

$$\frac{\partial h_i(p, v(p,m))}{\partial p_i} = -w(p)m\left[\frac{-2}{w(p)^3}\left(\frac{\partial w}{\partial p_i}\right)^2 + \frac{1}{w(p)^2}\frac{\partial^2 w}{\partial p_i^2}\right]$$

$$= \frac{m}{w(p)}\left[\frac{2}{w(p)}\left(\frac{\partial w}{\partial p_i}\right)^2 - \frac{\partial^2 w}{\partial p_i^2}\right]$$

Income effect:

$$-\frac{\partial x_i}{\partial m}\cdot x_i = \frac{\frac{\partial w(p)}{\partial p_i}}{w(p)}\cdot\left[\frac{-\frac{\partial w}{\partial p_i}\cdot m}{w(p)}\right] = -\frac{\left(\frac{\partial w}{\partial p_i}\right)^2 m}{w(p)^2}$$

Let's check to make sure that the substitution and income effects add up to the total effect:

$$\frac{m}{w}\left[\frac{2}{w}\left(\frac{\partial w}{\partial p_i}\right)^2 - \frac{\partial^2 w}{\partial p_i^2}\right] - \left(\frac{\partial w}{\partial p_i}\right)^2\frac{m}{w^2} = \frac{m}{w^2}\left(\frac{\partial w}{\partial p_i}\right)^2 - \frac{m}{w}\frac{\partial^2 w}{\partial p_i^2}$$

This is equal to the total effect $\frac{\partial x_i}{\partial p_i}$ we computed earlier.

(c) We can aggregate over consumers $i=1 \dots n$ if all of their indirect utility functions are in the Gorman form: $V_i(p,m) = a_i(p) + b(p)\,m$.
The indirect utility function $V(p,m) = w(p)\,m$ is in the Gorman form with $a_i(p)=0$. So aggregation is possible. However, all consumers have to have the same $w(p)$ function because $b(p)$ cannot vary over $i$. Since $a_i(p)=0$ for everyone, this implies that all consumers have identical preferences. The aggregate indirect utility is $V(p,M) = w(p)\,M$ where $M = \sum_i m_i$ is total income. In this case the market demand functions obtained by aggregating all the individual Marshallian demands will have the standard properties of any Marshallian demands: Slutsky ~~matrix~~ equation, symmetric and negative semi-definite substitution matrix, etc.

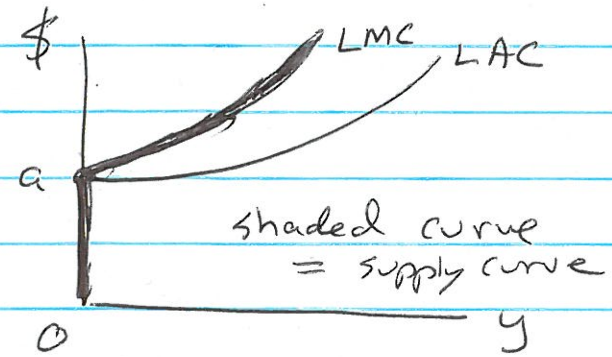
5 (a) (i) no change in the output supply function of a typical firm. The output supply curve for a firm in the short run is the part of the MC curve above AVC, and zero output for prices below min AVC (if there are such prices). Neither MC nor AVC is affected by F, so there is no effect on the output supply curve.

(ii) no effect on the market equilibrium price. Since the output supply functions for individual firms do not change, the market supply function $S(p)$ does not change. F has no effect on market demand $D(p)$ so the equilibrium price where $S(p) = D(p)$ also does not change.

(iii) All firms have lower profit. Suppose firm $i$ operates where $p = c'(y_i^*)$ with $p \geq \frac{c_v(y_i^*)}{y_i^*} = AVC(y_i^*)$. Then its profit is $py_i^* - c_v(y_i^*) - F$

The increase in F does not change $y_i^*$ but does reduce profit. Alternatively suppose firm $i$ shuts down so $y_i = 0$. Then its profit is just $-F$. Again this falls (the firm's loss is bigger) if F rises.

(b) $LAC = a + by^2 \Rightarrow c(y) = ay + by^3 \Rightarrow LMC = a + 3by^2$

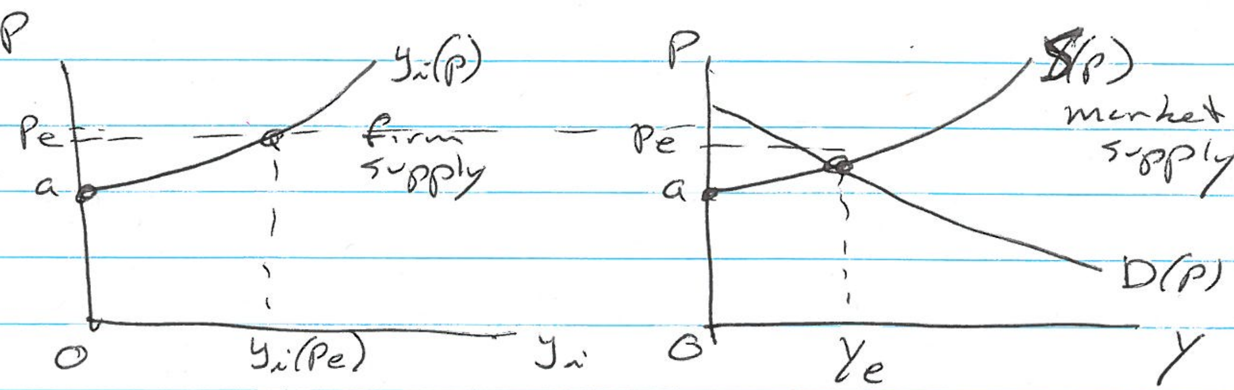



For $p \leq a$, $y(p) = 0$.

For $p > a$, $p = c'(y) = a + 3by^2$

$$\Rightarrow \frac{p-a}{3b} = y^2 \Rightarrow y(p) = \sqrt{\frac{p-a}{3b}}$$

(i)




We have $S(p) = \sum_i y_i(p)$. If $D(p)$ has a vertical intercept at or below a, there is zero output in equilibrium. If $D(p)$ has a vertical intercept above a, the equilibrium quantity is positive and equilibrium price is above a as shown.

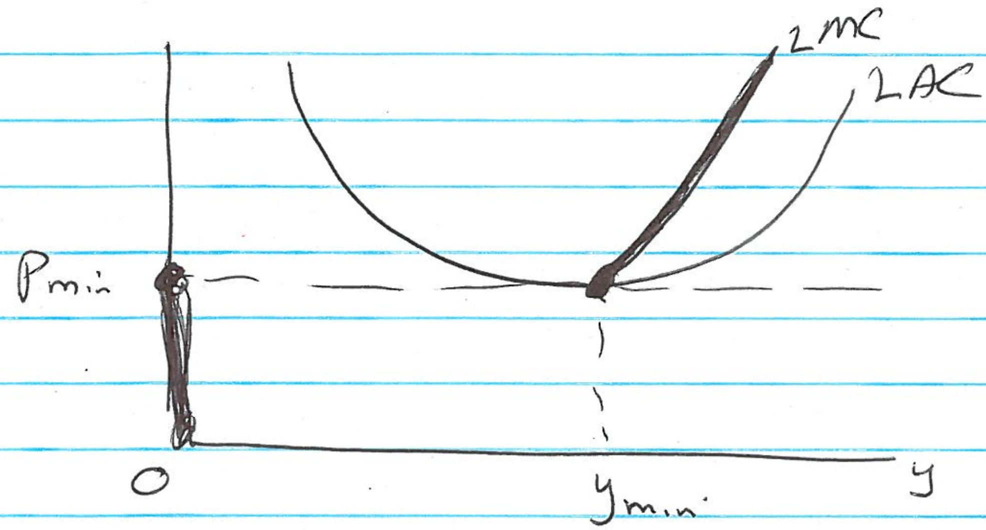
(ii) With free entry we have to think about profit. We have $p = c'(y_e) > \frac{c(y_e)}{y_e} = AC$ whenever $y_e > 0$. The fact that $P > AC \Rightarrow$ profit is positive so more firms enter. This flattens out $S(p)$. In the limit the result is an equilibrium price $p^* = a$, an equilibrium market quantity $D(p^*) = D(a)$ and each firm producing a negligible amount ($y_i \to 0$).

(c) $AC = A + By + Cy^2 \Rightarrow C(y) = Ay + By^2 + Cy^3$
$\Rightarrow MC = A + 2By + 3Cy^3$

LAC is quadratic and U-shaped: $LAC' = B + 2Cy = 0$
$\Rightarrow y_{min} = \frac{-B}{2C} > 0$. The level of AC at $y_{min}$ is

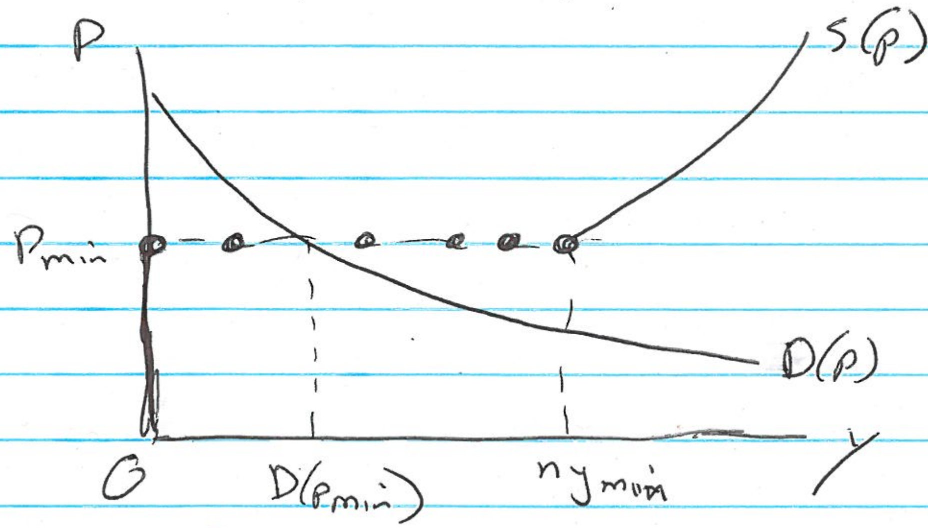
$$A + By_{min} + C(y_{min})^2 = A - \frac{B^2}{2C} + \frac{CB^2}{4C^2} = A - \frac{B^2}{4C}$$

Call this $P_{min}$ (see graph)




LMC intersects LAC at $y_{min}$ and is above LAC when LAC rises. The firm's supply function is the heavy shaded part of LMC, and the part of the vertical axis at and below $p_{min}$.

There could be a problem with non-existence of market equilibrium. Suppose at $p_{min}$ the market demand $D(p_{min})$ is positive but less than $ny_{min}$




Unless $D(p_{min})$ is some integer multiple of $y_{min}$, we have a problem: each firm is indifferent between $y_i = 0$ and $y_{min}$

But no firm is willing to produce any other output at $p_{min}$ so ~~we~~ can't get each individual firm to max profit and at the same time have output add up to $D(p_{min})$